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# Static Solutions of the Brans–Dicke Equations

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*Abstract*

It is shown that, given any static solution of the Einstein vacuum equations, a corresponding family of static vacuum solutions of the Brans–Dicke equations can be written down by inspection. Spherically and axially symmetric fields are considered explicitly. It is demonstrated how some solutions of the Brans–Dicke equations may be obtained without having to solve any field equations explicitly at all.

## 1. *Introduction*

In Einstein's theory of gravitation the Einstein tensor

$$G_{kl} := \tfrac{1}{2} g_{kl} R - R_{kl} \tag{1.1}$$

where†

$$R_{kl} := 2\Gamma^m{}_{k[m,l]} + 2\Gamma^n{}_{k[m}\Gamma^m{}_{l]n} \tag{1.2}$$

is the Ricci tensor of a four-dimensional normal-hyperbolic Riemann space $V_4$, is identified directly with the tensor $T_{kl}$ which characterises the distribution of stresses, momentum and energy. In other words the field equations of the theory are

$$G_{kl} = 8\pi\kappa T_{kl} \tag{1.3}$$

where $\kappa$ is a constant; and it is an immediate consequence of these that

$$T^{kl}{}_{;l} = 0 \tag{1.4}$$

From a formal point of view the scalar-tensor theory of Brans & Dicke (1961) modifies (1.3) in the following way: (i) the coupling constant $\kappa$ becomes a variable coupling paramenter $\phi^{-1}$; (ii) the energy-momentum tensor of the scalar field $\phi$ is included in $T_{kl}$, so that this field also appears as

† In the 'equations' $A := B$, $A =: B$ the colon serves to emphasise that these are defining relations for $A$ and $B$ respectively.

a source of the gravitational field; (iii) further terms are added on the right so as to ensure that (1.4) continues to hold. Explicitly, (1.3) is replaced by†

$$G_{kl} = 8\pi\phi^{-1} T_{kl} + \omega\phi^{-2}(\phi_{;k}\phi_{;l} - \tfrac{1}{2}g_{kl}\phi_{;m}\phi^{;m}) + \phi^{-1}(\phi_{;kl} - g_{kl}\Box\phi) \quad (1.5)$$

where $\omega$ is a constant, and the speed of light has been taken as unity. In addition there is a field equation for $\phi$:

$$(3 + 2\omega)\Box\phi = 8\pi T_k^{\ k} \quad (1.6)$$

(1.5) and (1.6) can be derived from a variational principle but this may be left aside here.

Now, the problem of obtaining exact solutions of these equations is evidently a formidable task. One is most likely to succeed under specialised circumstances, and the case of static vacuum fields at once suggests itself. Even then the explicit equations, as exemplified by equations (3.8–12) of Brans (1962) for the spherically symmetric case, are quite complicated. It would therefore be helpful to have some means of obtaining solutions—even if not the most general—by inspection from known solutions of simpler field equations (relating to the same circumstances). I show here that this may indeed be done.

To describe the main result obtained, let the indices $a$, $b$, $c$ take values only from the range 1 to 3. The metric $g_{kl}$ will be called static if

$$g_{kl,4} = 0 \qquad \textit{and} \qquad g_{a4} = 0 \quad (1.7)$$

and, granted (1.7), write $g_{kl} \equiv (g_{ab}, e^{2\psi})$, in an obvious notation. Then I prove‡ in Section 2 the following:

*Theorem*:

If $(g_{ab}, e^{2\psi})$ is a static vacuum solution of Einstein's equations then

$$g_{kl} = (e^{2\xi\psi} g_{ab}, e^{2\eta\psi}), \qquad \phi = e^{\zeta\psi} \quad (1.8)$$

is a static vacuum solution of the Brans–Dicke equation provided

$$\xi + \eta + \zeta = 1 \qquad \textit{and} \qquad \eta^2 + \eta\zeta + (\tfrac{1}{2}\omega + 1)\zeta^2 = 1 \quad (1.9)$$

Evidently (1.8) in fact represents a one-parameter *family* of solutions. In particular, under conditions of spherical symmetry this coincides with that given by Brans (1962); so that the latter is generated very simply by the well-known Schwarzschild metric (Section 3a). The general result quoted above may likewise be used in the context of the class of axially symmetric solutions of Weyl (Section 3b), and of other known solutions. In particular one may start with a galilean metric and still arrive at non-trivial solutions

† The apparent reversal of sign of the left-hand member of (1.5)—as compared with equation (11) of Brans & Dicke (1961)—occurs because these authors write $-R_{kl}$ where $R_{kl}$ has been written in (1.2).

‡ The work is closely related to Buchdahl (1959). However, superficially at any rate, the field equations there differ from those considered now, and the final result is attained by a different method.

of the Brans–Dicke equations; yet in this process one never needs to solve (in the explicit sense) any field equations at all.

## 2. *Proof of the Theorem*

(a) Writing $\phi =: e^{\theta}$ the equations under consideration reduce to

$$G_{kl} = \theta_{;kl} + (\omega + 1)\,\theta_{;k}\,\theta_{;l} - \tfrac{1}{2}\omega g_{kl}\,\theta_{;m}\,\theta^{;m} \tag{2.1}$$

$$\Box\,\theta + \theta_{;m}\,\theta^{;m} = 0 \tag{2.2}$$

where it has been assumed that $2\omega + 3 \neq 0$. Let $g_{kl}$ now be written in the form

$$g_{kl} =: (e^{2p}\,\bar{g}_{ab}, e^{2q}) \tag{2.3}$$

The $\bar{g}_{ab}$ may be looked upon as the components of the metric tensor of a (negative definite) three-dimensional Riemann space $\bar{V}_3$. Covariant differentiation in this $\bar{V}_3$ will be indicated by indices preceded by a colon; and its Ricci tensor, scalar curvature, and so on will be distinguished by bars. Then (Buchdahl, 1954)

$$G_{ab} = \bar{G}_{ab} - p_{:ab} - q_{:ab} + p_{:a}p_{:b} + p_{:a}q_{:b} + p_{:b}q_{:a} - q_{:a}q_{:b} + \bar{g}_{ab}(p_{:c}{}^{c} + q_{:c}{}^{c} + q_{:c}q^{:c}) \tag{2.4}$$

$$G_{44} = e^{2q-2p}(\tfrac{1}{2}\bar{R} + 2p_{:c}{}^{c} + p_{:c}p^{:c}) \tag{2.5}$$

Further,

$$\theta_{;ab} = \theta_{:ab} - p_{:a}\,\theta_{:b} - p_{:b}\,\theta_{:a} + \bar{g}_{ab}\,p_{:c}\,\theta^{:c} \tag{2.6}$$

$$\theta_{;44} = e^{2q-2p}\,q_{:c}\,\theta^{:c} \tag{2.7}$$

(b) Set

$$p = \xi\psi, \qquad q = \eta\psi, \qquad \theta = \zeta\psi \tag{2.8}$$

where $\xi$, $\eta$, $\zeta$ are constants. Then, using (2.6) and (2.7), equation (2.2) becomes

$$\zeta[\psi_{:c}{}^{c} + (\xi + \eta + \zeta)\,\psi_{:c}\,\psi^{:c}] = 0 \tag{2.9}$$

By the same token, drawing also upon (2.5), the (4,4)-member of the equations (2.1) becomes

$$\tfrac{1}{2}\bar{R} = -2\xi\psi_{:c}{}^{c} + (\eta\zeta - \xi^2 - \tfrac{1}{2}\omega\zeta^2)\,\psi_{:c}\,\psi^{:c} \tag{2.10}$$

Now, let the space whose metric tensor is $(\bar{g}_{ab}, e^{2\psi})$ be Ricci-flat, i.e. $G_{kl} = 0$. Setting $p = 0$, $q = \psi$ in (2.4) and (2.5), it follows that

$$\bar{G}_{ab} = \psi_{:ab} + \psi_{:a}\psi_{:b} - \bar{g}_{ab}(\psi_{:c}{}^{c} + \psi_{:c}\psi^{:c}) \tag{2.11}$$

and

$$\bar{R} = 0 \tag{2.12}$$

By transvection of (2.11) with $\bar{g}^{ab}$ one concludes that

$$\psi_{:c}{}^{c} + \psi_{:c}\psi^{:c} = 0 \tag{2.13}$$

Granted that $\zeta \neq 0$ and that $\psi$ is not constant, consistency of (2.9) with (2.13) requires that

$$\xi + \eta + \zeta = 1 \tag{2.14}$$

Further, consistency of (2.10) with (2.11) and (2.13) requires that

$$\eta\zeta - \xi^2 + 2\xi - \tfrac{1}{2}\omega\zeta^2 = 0 \tag{2.15}$$

which, because of (2.14), may be rewritten in the form

$$\eta^2 + \eta\zeta + (1 + \tfrac{1}{2}\omega)\,\zeta^2 = 1 \tag{2.16}$$

One still has to consider the remaining equations of (2.1). Both members of the $(k,4)$-components vanish identically because the field is static. As regards the $(a,b)$-components, one need only insert (2.8), (2.11) and (2.13) in (2.4) on the left and (2.6) on the right to conclude that the equation

$$(\xi + \eta + \zeta - 1)\,\psi_{:ab} - [\xi^2 + 2\xi\eta - \eta^2 + 2\xi\zeta - (\omega + 1)\,\zeta^2 + 1]\,\psi_{:a}\psi_{:b}$$
$$+ \bar{g}_{ab}(\xi\zeta - \eta^2 - \tfrac{1}{2}\omega\zeta^2 - \xi - \eta)\,\psi_{:c}\psi^{:c} = 0$$

needs to be satisfied. However, on account of (2.14) and (2.16) its left-hand member vanishes identically. This means that (2.1) and (2.2) are all satisfied, and the theorem stated in Section 1 is thus proved.

## 3. *Spherical and Axial Symmetry*

(a) Under conditions of spherical symmetry the solution of Einstein's vacuum equation is the Schwarzschild exterior metric. In canonical coordinates this is

$$ds^2 = -(1 - 4b/r_1)^{-1}\,dr_1{}^2 - r_1{}^2\,d\Omega^2 + (1 - 4b/r_1)\,dt^2 \tag{3.1}$$

where $d\Omega^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2$ and $b$ is a constant. If the coordinates are isotropic one has equivalently

$$ds^2 = -(1 + b/r)^4\,(dr^2 + r^2\,d\Omega^2) + (1 - b/r)^2\,(1 + b/r)^{-2}\,dt^2 \tag{3.2}$$

In principle it is of course a matter of indifference whether one generates solutions of the Brans–Dicke equations from (3.1) or from (3.2), or, for that matter, from any other static metric derived from them by transformations of coordinates, unless one imposes a coordinate condition upon them. In particular the Brans–Dicke solutions will immediately refer to isotropic coordinates only if one starts from (3.2). In fact, according to (1.8) they are

$$g_{11} = g_{22}/r^2 = g_{33}/r^2\sin^2\alpha = -(1 - b/r)^{2\xi}(1 + b/r)^{4-2\xi}$$
$$g_{44} = (1 - b/r)^{2\eta}(1 + b/r)^{-2\eta}, \qquad \phi = (1 - b/r)^{\zeta}\,(1 + b/r)^{-\zeta} \tag{3.3}$$

Inspection shows that this is substantially form I of the solution as given in the Appendix of Brans (1962), granted the identifications

$$\xi = 1 - (C + 1)/\lambda, \qquad \eta = 1/\lambda, \qquad \zeta = C/\lambda \tag{3.4}$$

[Brans' constants $e^{\alpha_0}$, $e^{\beta_0}$, $\phi_0$ can be made to appear by making the changes of scale $r \to re^{\beta_0}$, $t \to te^{\alpha_0}$ of the coordinates $r$, $t$ and writing $\phi/\phi_0$ in place of $\phi$; then also $b = Be^{\beta_0}$.] (3.4) is obviously consistent with (2.14), whilst (2.16) requires that

$$\lambda^2 = 1 + C + (1 + \tfrac{1}{2}\omega)\, C^2 \tag{3.5}$$

and this equation serves as the definition of $\lambda$ in Brans' work. (The other forms of solution can be obtained from form I—by carrying out limiting processes in the case of forms III and IV—but this is not the place to spell this out in detail.)

(b) Something new is obtained from the axially symmetric solutions of the Einstein equations due to Weyl and Levi-Civita (Robertson & Noonan, 1968). With $r := x^1$, $z := x^2$, $\alpha := x^3$, $t := x^4$ they are

$$ds^2 = -e^{-2\psi}[e^{2\gamma}(dr^2 + dz^2) + r^2\, d\alpha^2] + e^{2\psi}\, dt^2 \tag{3.6}$$

where $\psi$ is any solution of the two-dimensional flat-space Laplace equation

$$r^{-1}(r\psi_{,1})_{,1} + \psi_{,22} = 0 \tag{3.7}$$

and $\gamma$ is obtained from the equation

$$d\gamma = r[(\psi_{,1})^2 - (\psi_{,2})^2]\, dr + 2r\psi_{,1}\,\psi_{,2}\, dz \tag{3.8}$$

the integrability conditions on which are satisfied as a consequence of the field equations. The induced family of solutions of the Brans–Dicke equations is

$$\begin{aligned} ds^2 &= -e^{-2(\eta+\zeta)\psi}[e^{2\gamma}(dr^2 + dz^2) + r^2\, d\alpha^2] + e^{2\eta\psi}\, dt^2 \\ \phi &= e^{\zeta\psi} \end{aligned} \tag{3.9}$$

subject to $\eta$ and $\zeta$ satisfying the condition (2.16).

## 4. *Flat Space as Starting Point*

Einstein's vacuum equations are trivially satisfied when the $V_4$ is flat. It is worth noting that, starting from a galilean metric, one can therefore generate solutions of the Brans–Dicke equations without formally ever solving any equations at all. A simple example will illustrate this remark.

Let

$$ds^2 = -d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 + d\bar{t}^2 \tag{4.1}$$

to begin with. The coordinate transformation

$$\bar{x} = x\cosh t, \qquad \bar{y} = y, \qquad \bar{z} = z, \qquad \bar{t} = x\sinh t \tag{4.2}$$

changes this into

$$ds^2 = -dx^2 - dy^2 - dz^2 + x^2\, dt^2 \tag{4.3}$$

This then induces the following family of solutions of the Brans–Dicke equations:

$$ds^2 = -x^{2\xi}(dx^2 + dy^2 + dz^2) + x^{2\eta}\, dt^2, \qquad \phi = x^{\zeta} \tag{4.4}$$

More general solutions may be obtained in this way. It is best first to generate solutions of the Einstein equations by a succession of reciprocal transformations (Buchdahl, 1954) and appropriately chosen coordinate transformations, and then to write down the induced family of solutions of the Brans–Dicke equations.

Finally it should be remarked that instead of considering metrics which are static in the sense of (1.7), one may throughout contemplate metrics which are 'static with respect to the coordinate $x^s$', meaning that $g_{kl,s} = 0$ and $g_{ks} = 0$ $(k \neq s)$. The work of Section 2 remains essentially unchanged: one need only write $g_{ss}$ in place of $g_{44}$ and take the range of the indices $a, b, c$ to exclude the value $s$. By way of example, take the obviously flat metric

$$ds^2 = -t^2\,dx^2 - dy^2 - dz^2 + dt^2$$

in place of (4.3). This is static with respect to $x$, and it therefore induces the solution

$$ds^2 = -t^{2\eta}\,dx^2 - t^{2\xi}(dy^2 + dz^2) + t^{2\xi}\,dt^2, \qquad \phi = t^{\zeta} \tag{4.5}$$

of the Brans–Dicke equations. In any physical interpretation the fields are, however, now time-dependent.

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